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Asymptotic Distributions of Continuous-Time Random Walks: A Probabilistic Approach

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We provide a systematic analysis of the possible asymptotic distributions of one-dimensional continuous-time random walks (CTRWs) by applying the limit theorems of probability theory. Biased and unbiased walks of coupled and decoupled memory are considered. In contrast to previous work concerning decoupled memory and Lévy walks, we deal also with arbitrary coupled memory and with jump densities asymmetric about its mean, obtaining asymmetric Lévy-stable limits. Suprisingly, it is found that in most cases coupled memory has no essential influence on the form of the limiting distribution. We discuss interesting properties of walks with an infinite mean waiting time between successive jumps.

KEY WORDS: Random walks; coupled memory; asymptotic distributions; Lévy-stable distributions.

1. INTRODUCTION

A continuous-time random walk (CTRW) is a walk with random waiting times T_i between successive random jumps R_i. This notion was introduced in a paper of Montroll and Weiss.[1] Since then it has been studied extensively and applied, for instance, to fully developed turbulence,[3,4] transport in disordered or fractal media,[5-7] intermittent chaotic systems,[8,10] and relaxation phenomena.[11,12] The common feature of these applications is that they exhibit anomalous diffusion, which is manifested by a nonlinear time dependence of the mean square displacement. The signature of the anomalous diffusion is also a non-Gaussian asymptotic distribution

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(propagator, diffusion front) of distance R_t reached up to a large time t by a particle initially at the origin. Usually, the analysis of the asymptotic distribution is based on a formal expression for the Fourier–Laplace transform of R_t, but a useful, explicit inversion formula has been provided only under some restrictive assumptions on spatiotemporal coupling. In general, the ith jump R_i of CTRW depends on its waiting time T_i in an arbitrary way, yielding both decoupled and a class of various coupled memory CTRWs. In most analyses[13–15] the asymptotic distribution of R_t was derived for independent (i.e., decoupled) R_i and T_i. Also, the specific case of the Lévy walk has been analyzed.[16–18]

In this paper we present a straightforward probabilistic approach in terms of random variables R_i and T_i, applying limit theorems of probability theory directly. This allows us to:

1. Assume any dependence between the random jump R_i and its waiting time T_i, including the cases of Lévy walk and decoupled memory discussed in the literature.[13–18]

2. Consider random walks with continuous as well as discrete variables R_i and T_i.

3. Deal also with probability densities $f(r)$ of the jumps R_i which are asymmetric about the mean (or median if the mean does not exist), which is in contrast to previous work focusing only on the case of symmetric random walks.

4. Include in the considerations biased and unbiased walks, i.e., include jump densities $f(r)$ with an arbitrary mean, which we denote by μ.

The aim of this paper is to investigate the possible forms of asymptotic distributions of distance R_t reached by a particle up to moment t, as $t \to \infty$, under the most general conditions 1–4. This completes and unifies the results which have appeared in several papers.[13–26] The considerations presented below lead also to a derivation of the normalizing constants c_t (prefactors) in the limiting procedure which are very useful in computer simulations but have not been derived in explicit form before.

This paper is organized as follows. Section 2.1 contains the description of CTRW in terms of random variables R_i, T_i. Section 2.2 provides a detailed exposition of the parameters and distributions which are necessary in Table I to summarize the limiting behavior of R_t in different cases. Section 3 is devoted to the derivation of the results. In Section 3.3 and in the last part of Section 3.2 we discuss interesting properties of the case which assumes that the mean of the waiting times T_i is infinite.

2. ASYMPTOTIC BEHAVIOR

2.1. The CTRW as a Random Sum

Let us consider a continuous-time random walk described by means of two sequences of random variables R_i and T_i, jump distances and waiting time intervals, respectively, with a particle starting at point $R_0 \equiv 0$ at time $T_0 \equiv 0$. For simplicity we restrict our attention to one-dimensional walks. The first instantaneous jump of random length R_1 takes place after a random waiting time T_1, then the second instantaneous jump R_2 after time T_2, etc. (see Fig. 1). In general, the ith jump R_i may depend on its waiting time T_i in an arbitrary way, but the pair (R_i, T_i) is independent of the preceding and succeeding pairs of jumps and its waiting times (R_k, T_k). The special case when R_i is independent of T_i is called a decoupled memory CTRW, as opposed to the coupled one with R_i depending on T_i. It is evident that the model is entirely determined by $\psi(r, t)$, a two-dimensional joint probability density of the pair (R_i, T_i), the same for each $i \geqslant 1$. The function $\psi(r, t)$ can be any density on the half-plane $-\infty < r < \infty$, $t \geqslant 0$. The marginal densities $f(r)$ of R_i and $g(t)$ of T_i are

$$f(r) = \int_0^\infty \psi(r, t)\, dt, \qquad g(t) = \int_{-\infty}^\infty \psi(r, t)\, dr \tag{1}$$

When the function $f(r)$ does not contain the Dirac delta component then the jump R_i has a continuous distribution. If $f(r)$ is a linear combination

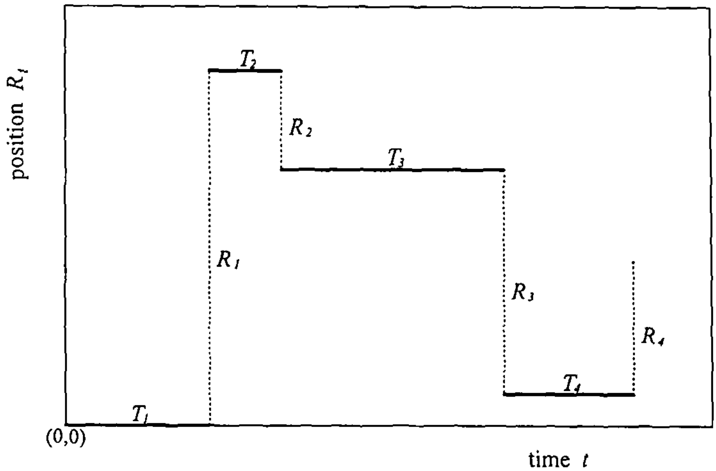

Fig. 1. Single realization of a continuous-time random walk.

of Dirac delta functions, a discrete (lattice) distribution of R_i is obtained.[2]

In this paper we will be interested in finding for large t the asymptotic distribution of position R_t of the particle at time t. The *probability density of* R_t will be denoted by $p_t(r)$. Define the random variable N_t as the number of jumps in the time interval $[0, t]$,

$$N_t = \max \left\{ k \colon \sum_{i=0}^{k} T_i \leqslant t \right\} \tag{2}$$

It is clear that position R_t of the particle at time t is equal to a random sum of N_t successive random jumps R_i,

$$R_t = \sum_{i=0}^{N_t} R_i \tag{3}$$

Note that formula (3) holds also for random walks when the waiting time intervals are nonrandom and take a constant value.

2.2. Summary of the Results

For the reader's convenience we summarize in Table I the results whose derivation is presented in Section 3. The constant $\tau \equiv \langle T_i \rangle$ denotes the *mean waiting time* and $\mu \equiv \langle R_i \rangle$ is the mean single jump distance (provided that it exists), called later the *bias*.

The parameters α, β take the constant values

$$\alpha = 2 \qquad \text{and} \qquad \beta = 0 \qquad \text{if} \qquad \langle R_i^2 \rangle \text{ is finite} \tag{4}$$

or, if $\langle R_i^2 \rangle$ is infinite, the parameters α, β describe the tails of the density $f(r)$ of the jump distances R_i [cf. Eq. (1)] as follows:

$$f(-r) + f(r) \sim b r^{-\alpha - 1} \qquad \text{and} \qquad \frac{f(r)}{f(-r) + f(r)} \to \frac{1+\beta}{2} \tag{5}$$

for $r \to \infty$, where b is a constant called later a *magnitude coefficient* of the density $f(r)$ and $0 < \alpha < 2$, $-1 \leqslant \beta \leqslant 1$. From condition (5) it may be concluded that

$$f(-r) \sim b \frac{1-\beta}{2} r^{-\alpha-1} \qquad \text{and} \qquad f(r) \sim b \frac{1+\beta}{2} r^{-\alpha-1}$$

[2] In the probabilistic literature only absolutely continuous distributions, in contrast to discrete and singular ones, are said to have a density function; see, for instance, ref. 9.

Table I. Asymptotic Behavior of the Displacement R_t for a CTRW[a]

$\alpha' > 1$, i.e., a finite mean waiting time τ	
$0<\alpha<1$, any median of R_i $1<\alpha<2$, $\mu=0$	$\lim_{t\to\infty} \Pr\left(\frac{R_t}{(t/\tau)^{1/\alpha} c_2} \leqslant r\right) = S_{\alpha,\beta}(r)$
$\alpha=2$, $\mu=0$	$\lim_{t\to\infty} \Pr\left(\frac{R_t}{(t/\tau)^{1/2} c_1} \leqslant r\right) = \Phi(r)$
$1<\alpha<2$, $\mu\neq 0, \alpha<\alpha'$	$\lim_{t\to\infty} \Pr\left(\frac{R_t-(t/\tau)\mu}{(t/\tau)^{1/\alpha} c_2} \leqslant r\right) = S_{\alpha,\beta}(r)$
$1<\alpha\leqslant 2$, $\mu\neq 0, \alpha>\alpha'$	$\lim_{t\to\infty} \Pr\left(\frac{R_t/\mu-(t/\tau)}{t^{1/\alpha'} c_3} < r\right) = S_{\alpha',-1}(r)$
$1<\alpha<2$, $\mu\neq 0, \alpha=\alpha'$	$\lim_{t\to\infty} \Pr\left(\frac{R_t-(t/\tau)\mu}{(t/\tau)^{1/\alpha} c_5} \leqslant r\right) = S_{\alpha,\beta_1}(r)$
$\alpha=2$, $\mu\neq 0, \alpha=\alpha'$	$\lim_{t\to\infty} \Pr\left(\frac{R_t-(t/\tau)\mu}{(t/\tau)^{1/2} c_4} \leqslant r\right) = \Phi(r)$
$0<\alpha'<1$, i.e., an infinite mean waiting time	
$1<\alpha\leqslant 2$, $\mu\neq 0$	$\lim_{t\to\infty} \Pr\left(\frac{R_t/\mu}{t^{\alpha'} c_6} < r\right) = H_{\alpha'}(r)$
$0<\alpha<1$, any median of R_i $1<\alpha\leqslant 2$, $\mu=0$	$\lim_{t\to\infty} \Pr\left(\frac{R_t}{t^{\alpha'/\alpha} c_7} \leqslant r\right) = U(r)$

[a] The notation is explained in Section 2.2. In all cases coupled as well as decoupled memories are assumed, except in the last one, where the limiting distribution $U(r)$ only for the decoupled memory is obtained [cf. Eqs. (36)–(38)].

which shows that the parameter β governs the asymmetry of the tails of $f(r)$. Note that $\beta=0$ is equivalent, in the limit, to symmetry of the tails. Equations (4) and (5) are equivalent to the statement that the density $f(r)$ belongs to the *domain of attraction*[2,9] of the standard Lévy-stable density $s_{\alpha,\beta}(r)$ with $0<\alpha\leqslant 2$ and $-1\leqslant\beta\leqslant 1$ [see Eq. (15) below]. Observe that the density $s_{2,\beta}(r)=s_{2,0}(r)$ for any β and is equal to the Gaussian one with mean 0 and variance 2. In the following we exclude the case $\alpha=1$, $\beta\neq 0$ because densities $s_{1,\beta}(r)$ with $\beta\neq 0$ have peculiar properties and some authors even do not consider them to be Lévy-stable (cf. ref. 2, p. 23).

The parameter α' takes the constant value

$$\alpha'=2 \qquad \text{if} \quad \langle T_i^2\rangle \quad \text{is finite} \tag{6}$$

or, if $\langle T_i^2 \rangle$ is infinite, the parameter α' describes the tail of the density $g(t)$ of the waiting time interval T_i [cf. Eq. (1)] as follows:

$$g(t) \sim b' t^{-\alpha' - 1} \tag{7}$$

for $t \to \infty$, where b' is the magnitude coefficient of $g(t)$ and $0 < \alpha' < 2$. Equations (6) and (7) are equivalent to the statement that the density $g(t)$ belongs to the domain of attraction of the standard Lévy-stable density $s_{\alpha', 1}(t)$ (note that the second parameter equals 1 because $T_i \geqslant 0$) when $0 < \alpha' < 2$ or the Gaussian density $s_{2,0}(t)$ when $\alpha' = 2$.

The *normalizing constants* $c_1, \ldots, c_7$ (prefactors) given in Section 3 are functions of the parameters α, α' and the magnitude coefficients b, b' introduced in Eqs. (4–7).

The distribution function $S_{\alpha,\beta}(r)$ of the standard Lévy-stable density $s_{\alpha,\beta}(r)$ [cf. Eq. (15) below] equals

$$S_{\alpha,\beta}(r) = \int_{-\infty}^{r} s_{\alpha,\beta}(x)\, dx \tag{8}$$

Moreover,

$$\Phi(r) = (2\pi)^{-1/2} \int_{-\infty}^{r} \exp(-x^2/2)\, dx \tag{9}$$

denotes the Gaussian law with mean 0 and variance 1,

$$H_{\alpha'}(r) = 1 - S_{\alpha',1}\left(\frac{1}{r^{1/\alpha'}}\right), \quad r > 0 \qquad \text{and} \qquad H_{\alpha'}(r) = 0, \quad r \leqslant 0 \tag{10}$$

denotes the "inverse" Lévy-stable distribution function [cf. also Eq. (34)] and $U(r)$ is the distribution function defined in the integral formula (38) below.

For comparison, we list, using our notation the cases which have been investigated in the literature:

1. Tunaley[13]: $\alpha = 2$, $\beta = 0$, $\mu = 0$ or $\mu \neq 0$ for decoupled memory.

2. Shlesinger, Klafter, and Wong[14]: $0 < \alpha \leqslant 2$, $\beta = 0$, $\mu = 0$ for decoupled memory.

3. Weissman, Weiss, and Havlin[15]: $0 < \alpha \leqslant 2$, $\beta = 0$, $0 < \alpha' < 1$, $\mu = 0$ or $\mu \neq 0$ for decoupled memory.

4. Zumofen, Klafter, and Blumen[16]: $\alpha > 1$, $\beta = 0$, $\mu = 0$. They consider the Lévy walk with the parameter $\nu = 1$, i.e., coupled memory of a specific type.

5. Mantegna[17]: Monte Carlo simulations of the case $\alpha' > 1$, $\beta = 0$, $\mu = 0$ for Lévy walks with the parameter $\nu > 1/2$.

6. Araujo, Havlin, Weiss, and Stanley[18]: $1 < \alpha' < 2$, $\beta = 0$, $\mu = 0$. They consider the generalized Lévy walk with the parameter $\nu = 1$ or, in other words, the continuous-time generalization of the persistent random walk, i.e., coupled memory of a specific type.

7. See also Kotulski[30]: $0 < \alpha \leqslant 2$, $-1 \leqslant \beta \leqslant 1$, an arbitrary μ; in which a generalization of a Lévy walk allowing for an arbitrary bias and asymmetric tails of jump density $f(r)$ is considered.

3. DERIVATION OF RESULTS

3.1. Unbiased Walks with a Finite Mean Waiting Time

In this subsection we assume that:

(i) The mean jump distance $\mu = \langle R_i \rangle \equiv \int_{-\infty}^{\infty} r f(r)\, dr = 0$ (with the integral absolutely convergent) if $1 < \alpha \leqslant 2$, while if $0 < \alpha < 1$, the mean does not exist and we assume the median of R_i to be arbitrary [if $\alpha = 1$, we deal only with symmetric $f(r)$ with a median at 0].

(ii) The mean waiting time $\tau = \langle T_i \rangle$ is finite, i.e., $\alpha' > 1$.

Thus the density $f(r)$ of R_i is possibly asymmetric about its mean μ, which is equal to 0, or, when $\langle R_i \rangle$ does not exist, $f(r)$ may be asymmetric about its median, which may take any value.

Consider first the degenerate case $T_i \equiv \tau_0 = \text{const}$. Then in Eq. (3) we sum the nonrandom number $N_t = \lfloor t/\tau_0 \rfloor$ of independent jumps R_i, hence standard theorems about the limiting distribution of such a sum can be applied.[2,9] Provided $\langle R_i^2 \rangle$ is finite, the central limit theorem assures the convergence of normed R_t to the Gaussian law; when $\langle R_i^2 \rangle$ is infinite, the theory of P. Lévy gives the convergence of appropriately normed R_t to the Lévy-stable law.

Now, returning to CTRW, allow T_i to be random. We find the asymptotic value of N_t using renewal theory. The fundamental renewal theorem[9] states that $\lim_{t \to \infty}(\langle N_t \rangle / t) = 1/\tau$; also the strong version can be proved, namely

$$\lim_{t \to \infty} \frac{N_t}{(t/\tau)} = 1 \tag{11}$$

with probability 1. Hence, for large t, the number N_t of jumps in the time interval $[0, t]$ equals approximately t/τ and the random sum of N_t jumps R_i has the same limiting distribution as the nonrandom sum of t/τ jumps

R_i, so the above-mentioned limit theorems are still valid. The rigorous proof can be found in Wittenberg[19] or, in a more abstract setup, Csorgo and Rychlik[28] or Aldous.[29] Now, under assumptions (i), (ii), we formulate the limit theorems for CTRW in full extent.

If the second moment of R_i is finite, i.e., $\alpha = 2$, then

$$\Pr((t/\tau)^{-1/2} c_1^{-1} R_t \leqslant r) \to \Phi(r) \tag{12}$$

as $t \to \infty$, where $\Phi(r)$ is the Gaussian law defined in Eq. (9) and c_1^2 is the variance of R_i, that is, $c_1^2 = \mathrm{Var}(R_i) = \langle R_i^2 \rangle - \langle R_i \rangle^2$. In other words, for large t, the density $p_t(r)$ of R_t is approximately equal to a Gaussian density with mean 0 and standard deviation $(t/\tau)^{1/2} c_1$, that is, the standard Gaussian density $\phi(r) = (2\pi)^{-1/2} \exp(-r^2/2)$ with a scale elongation $(t/\tau)^{1/2} c_1$. Thus Eq. (12) can be written as

$$p_t(r) \approx \frac{1}{(t/\tau)^{1/2} c_1} \phi\left(\frac{r}{(t/\tau)^{1/2} c_1}\right) \tag{13}$$

If $\langle R_i^2 \rangle$ is infinite, we assume condition (5) to hold, which leads to

$$\Pr((t/\tau)^{-1/\alpha} c_2^{-1} R_t \leqslant r) \to S_{\alpha,\beta}(r) \equiv \int_{-\infty}^{r} s_{\alpha,\beta}(x)\, dx \tag{14}$$

as $t \to \infty$, with c_2 given by Eq. (16). The function $s_{\alpha,\beta}(r)$ is the standard Lévy-stable density with parameters α, β, which is defined by its characteristic function

$$\hat{s}_{\alpha,\beta}(u) \equiv \int_{-\infty}^{\infty} e^{iur} s_{\alpha,\beta}(r)\, dr = \exp\{-|u|^\alpha [1 - i\beta\omega(u)]\} \tag{15}$$

where $\omega(u) = \mathrm{sign}(u) \tan(\pi\alpha/2)$ for $\alpha \neq 1$, and $\omega(u) = -\mathrm{sign}(u)(2/\pi) \ln |u|$ for $\alpha = 1$. The parameter α is called the *index of stability* and determines the tail exponent of the density $s_{\alpha,\beta}(r)$, while the second parameter β $(-1 \leqslant \beta \leqslant 1)$ governs its symmetry, namely if $\beta = 0$, then $s_{\alpha,\beta}(r)$ is symmetric. For the graphs of various Lévy-stable densities see refs. 2, 27 and 30. Note that by substituting $\alpha = 2$ and any β into Eq. (15) one obtains the Gaussian characteristic function with mean 0 and variance 2; hence one typically associates the Gaussian law with the index of stability $\alpha = 2$ and the parameter $\beta = 0$ because of its symmetry.

The normalizing constant c_2 in formula (14) is given by

$$c_2 = [b/(\alpha C_\alpha)]^{1/\alpha} \tag{16}$$

where b is the magnitude coefficient of $f(r)$ introduced in Eq. (5) and the constant C_α satisfies

$$C_\alpha = \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)}, \quad \alpha \neq 1 \qquad \text{or} \qquad C_\alpha = \frac{2}{\pi}, \quad \alpha = 1 \tag{17}$$

Let us remark that αC_α is the magnitude coefficient of the Lévy-stable density $s_{\alpha,\beta}(r)$ whose tails are determined by the relations (see Janicki and Weron,[2] p. 25)

$$s_{\alpha,\beta}(-r) \sim \alpha C_\alpha \frac{1-\beta}{2} r^{-\alpha-1}, \qquad s_{\alpha,\beta}(r) \sim \alpha C_\alpha \frac{1+\beta}{2} r^{-\alpha-1} \tag{18}$$

as $r \to \infty$, where $0 < \alpha < 2$ and $-1 \leqslant \beta \leqslant 1$.

One may rewrite formula (14) in terms of the density $p_t(r)$ of R_t or its characteristic function $\hat{p}_t(u)$: under condition (5), for large t, the density $p_t(r)$ approximately equals the standard Lévy-stable density $s_{\alpha,\beta}(r)$ with scale elongation $(t/\tau)^{1/\alpha} c_2$, that is,

$$p_t(r) \approx \frac{1}{(t/\tau)^{1/\alpha} c_2} s_{\alpha,\beta}\left(\frac{r}{(t/\tau)^{1/\alpha} c_2}\right) \tag{19}$$

or, equivalently, using the characteristic function

$$\hat{p}_t(u) \equiv \int_{-\infty}^{\infty} e^{iur} p_t(r)\, dr \approx \exp\{-|c_2 u|^\alpha (t/\tau)[1 - i\beta\omega((t/\tau)^{1/\alpha} c_2 u)]\} \tag{20}$$

where c_2 is defined in Eq. (16) and $\omega(u)$ in Eq. (15). These alternative forms of expressing the asymptotic behavior of R_t can be easily applied to each equation in Table I.

Let us indicate some interesting consequences of formula (14). The normalizing constant c_2 defined in Eq. (16) is the function of the magnitude coefficient b and the parameter α; therefore the limiting distribution $S_{\alpha,\beta}(r)$ of R_t depends on these constants together with the parameter β and mean waiting time τ, but *does not* depend on the type of coupling between R_i and T_i. Also, when $0 < \alpha < 2$, in contrast to the Gaussian case, the possible asymmetry of the limiting distribution is determined by the asymptotic quotient of the left and right tails of $f(r)$ which is specified by the parameter β in Eq. (5). The precise form of $f(r)$ inside any finite interval does not affect the parameters α and β. Surprisingly, in the case of infinite mean jump distance, i.e., when $0 < \alpha < 1$, formula (14) holds for every median of $f(r)$ [cf. (i) above], so adding a large nonrandom constant to each jump does not change the limiting behavior of R_t. Finally, we remark that the asymptotic distribution of R_t as $t \to \infty$ says nothing about the moments of R_t.

3.2. Biased Walks

In this subsection we assume that the mean jump distance $\mu \equiv \langle R_i \rangle$ exists and does not equal zero, that is, $1 < \alpha \leqslant 2$ and $\mu \neq 0$.

Case $\alpha' > 1$. This assumes a finite mean waiting time $\tau \equiv \langle T_i \rangle$. We apply Eqs. (14) and (3) with $(R_i - \mu)$ instead of R_i, obtaining as $t \to \infty$

$$\Pr\left(\frac{R_t - N_t \mu}{(t/\tau)^{1/\alpha} c_2} \leqslant r\right) \to S_{\alpha,\beta}(r) \tag{21}$$

This formula is unsatisfactory because $N_t \mu$ is a random variable. Using Eq. (11), one could replace N_t in Eq. (21) with its asymptotic value t/τ, but it is not always permissible. To see this more clearly, we need the limiting distribution of N_t. When $t \to \infty$ (cf. ref. 9, XI.5.6) one finds

$$\Pr\left(\frac{N_t - t/\tau}{t^{1/\alpha'} c_3} < r\right) \to S_{\alpha',-1}(r), \qquad 1 < \alpha' < 2 \tag{22}$$

where the normalizing constant $c_3 = (\alpha' C_{\alpha'} \tau / b')^{-1/\alpha'} \tau^{-1}$ with b' defined by Eq. (7) and $C_{\alpha'}$ by Eq. (17).

1. Suppose that $\alpha < \alpha' \leqslant 2$ and write

$$\frac{R_t - (t/\tau)\mu}{(t/\tau)^{1/\alpha} c_2} = \frac{R_t - N_t \mu}{(t/\tau)^{1/\alpha} c_2} + \mu \frac{N_t - t/\tau}{(t/\tau)^{1/\alpha} c_2} \tag{23}$$

The second term on the right-hand side of Eq. (23) tends in probability to 0 because of Eq. (22) together with the relation $1/\alpha' < 1/\alpha$. Thus applying Lemma 2, VIII.2, in ref. 9, we find the same limiting behavior of the remaining terms of Eq. (23), so formula (21) yields for $t \to \infty$

$$\Pr\left(\frac{R_t - (t/\tau)\mu}{(t/\tau)^{1/\alpha} c_2} \leqslant r\right) \to S_{\alpha,\beta}(r), \qquad \alpha < \alpha' \tag{24}$$

2. Conversely, let us suppose that $2 \geqslant \alpha > \alpha'$ and write similarly to Eq. (23), with α' replacing α,

$$\frac{R_t - (t/\tau)\mu}{\mu t^{1/\alpha'} c_3} = \frac{R_t - N_t \mu}{\mu t^{1/\alpha'} c_3} + \frac{N_t - t/\tau}{t^{1/\alpha'} c_3} \tag{25}$$

The first term of the right-hand side of Eq. (25) tends in probability to 0 because of Eq. (21) together with the relation $1/\alpha' > 1/\alpha$. Thus the remaining terms of Eq. (25) have the same limit; hence formula (22) yields for $t \to \infty$

$$\Pr\left(\frac{R_t/\mu - (t/\tau)}{t^{1/\alpha'} c_3} < r\right) \to S_{\alpha', -1}(r), \qquad \alpha > \alpha' \tag{26}$$

3. We deal with $\alpha = \alpha' = 2$ separately. Since $\langle R_i - (\mu/\tau)\, T_i \rangle = 0$, we can apply Eqs. (12) and (3) with the random variable $(R_i - (\mu/\tau)\, T_i)$ replacing R_i,[20] obtaining as $t \to \infty$

$$\Pr\left(\left(\frac{t}{\tau}\right)^{-1/2} c_4^{-1}\left(R_t - \frac{\mu}{\tau}\sum_{i=0}^{N_t} T_i\right) \leqslant r\right) \to \Phi(r) \tag{27}$$

where c_4^2 is the variance of the random variable $(R_i - (\mu/\tau)\, T_i)$, namely

$$c_4^2 = \operatorname{Var}\left(R_i - \frac{\mu}{\tau} T_i\right) = \operatorname{Var}(R_i) + \frac{\mu^2}{\tau^2}\operatorname{Var}(T_i) - 2\frac{\mu}{\tau}\operatorname{Cov}(R_i, T_i) \tag{28}$$

Using once again Lemma 2, VIII.2, of ref. 9, we may replace $\sum_{i=0}^{N_t} T_i$ with t in Eq. (27), because $t^{-1/2}(t - \sum_{i=0}^{N_t} T_i)$ tends in probability to 0 (see ref. 9, XI.3), obtaining as $t \to \infty$

$$\Pr\left(\frac{R_t - (t/\tau)\mu}{(t/\tau)^{1/2} c_4} \leqslant r\right) \to \Phi(r) \tag{29}$$

Let us remark that Eq. (29) also holds for $\mu = 0$, yielding then Eq. (12).

4. Suppose that $\alpha = \alpha' < 2$. The same procedure for Eq. (14) instead of Eq. (12) yields

$$\Pr\left(\frac{R_t - (t/\tau)\mu}{(t/\tau)^{1/\alpha} c_5} \leqslant r\right) \to S_{\alpha, \beta_1}(r) \tag{30}$$

where α, β_1 describe the domain of attraction of the random variable $(R_i - (\mu/\tau)\, T_i)$, and the normalizing constant c_5 is defined like c_2 in Eq. (14) with b replaced by the magnitude coefficient of $(R_i - (\mu/\tau)\, T_i)$.

We therefore arrive at some unexpected conclusions: the asymptotic behavior of R_t is determined by the limiting distribution of $\sum_{i=1}^{n} R_i$ if $\alpha \leqslant \alpha'$ and by the limiting distribution of N_t if $\alpha > \alpha'$. Note also that the normalizing constants c_4 and c_5 in Eqs. (29) and (30) are sensitive to the form of the correlation between R_i and T_i.

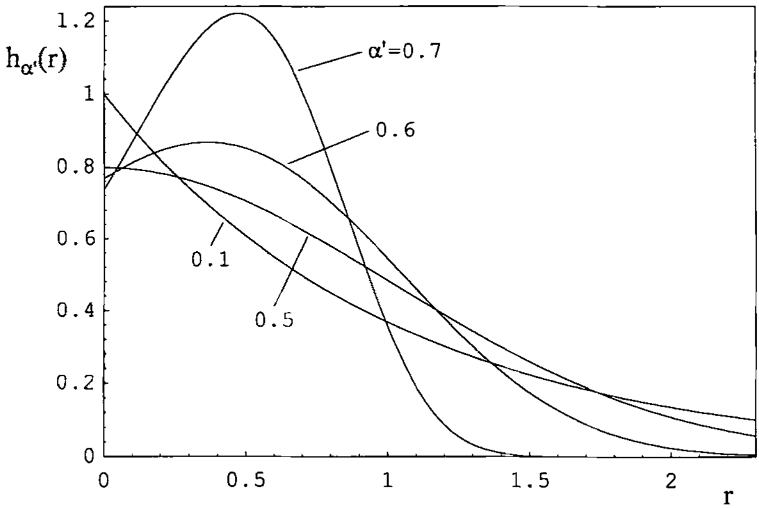

Fig. 2. Graphs of densities $h_{\alpha'}(r)$ for α' values equal to 0.1, 0.5, 0.6, and 0.7. Notice that the densities $h_{\alpha'}(r)$ are nonzero only at points $r > 0$.

Case $0 < \alpha' < 1$. This case relates to the infinite mean waiting time $\tau \equiv \langle T_i \rangle$. We need the limiting behavior of N_t (cf. ref. 9, XI.5.5). As $t \to \infty$

$$\Pr\left(\frac{N_t}{t^{\alpha'} c_6} < r\right) \to H_{\alpha'}(r), \qquad 0 < \alpha' < 1 \tag{31}$$

where $H_{\alpha'}(r)$ is the "inverse" Lévy-stable distribution defined in Eq. (10) and the normalizing constant $c_6 = \alpha' C_{\alpha'}/b'$ with b' and $C_{\alpha'}$ defined by Eqs. (7) and (17), respectively. Since $N_t \to \infty$ with probability 1, one obtains, using the strong law of large numbers,

$$\lim_{t \to \infty} \frac{R_t/\mu}{N_t} = 1 \tag{32}$$

with probability 1. Hence, by combining Eqs. (31) and (32) we conclude that (see also Smith,[21] p. 27)

$$\Pr\left(\frac{R_t/\mu}{t^{\alpha'} c_6} < r\right) \to H_{\alpha'}(r), \qquad 0 < \alpha' < 1 \tag{33}$$

The graphs of densities $h_{\alpha'}(r)$ of "inverse" Lévy-stable distributions $H_{\alpha'}(r)$ are given in Fig. 2. It is worth pointing out (see Feller,[9] XVII.6.10, or Zolotarev[27]) that for $1/2 \leqslant \alpha' < 1$ the density $h_{\alpha'}(r)$ is simply the cutoff stable density $s_{1/\alpha', -1}(r)$, so that it is restricted to the positive half-line only, namely

$$h_{\alpha'}(r) = (1/\alpha')\, s_{1/\alpha', -1}(r/c_8)/c_8, \quad r > 0 \qquad \text{for} \quad 1/2 \leqslant \alpha' < 1 \tag{34}$$

where the constant $c_8 = \cos(\pi\alpha'/2)\{-\cos[\pi/(2\alpha')]\}^{\alpha'}$ is the scale parameter. Note that the cutoff normalization term $1/\alpha'$ in Eq. (34) results from the relation $1/\alpha' = [1 - S_{1/\alpha', -1}(r/c_8)]^{-1}$ for $r = 0$. An interesting property of $h_{\alpha'}(r)$ is[27]

$$h_{\alpha'}(r) \to e^{-r} \qquad \text{as} \quad \alpha' \downarrow 0, \quad \text{for} \quad r > 0 \tag{35}$$

Let us indicate the qualitative differences between the case of biased walks with finite mean waiting time and the present case. With increasing time, the density of R_t in Eq. (33) is subject to a sublinear elongation of order $t^{\alpha'}$, but not to a shift proportional to time t as in Eqs. (24) and (26). Moreover, the limiting densities $h_{\alpha'}(r)$ are nonzero only on positive half-line and are discontinuous at $r = 0$.

3.3. Unbiased Walks with an Infinite Mean Waiting Time

In this subsection we assume that:

(i) The mean jump distance $\mu = \langle R_i \rangle \equiv \int_{-\infty}^{\infty} r f(r)\, dr = 0$ (with the integral absolutely convergent) if $1 < \alpha \leqslant 2$, while if $0 < \alpha < 1$, the mean does not exist and we assume the median of R_i to be arbitrary [if $\alpha = 1$, we deal only with symmetric $f(r)$ with a median at 0].

(ii) The mean waiting time $\tau = \langle T_i \rangle$ is infinite, i.e., $0 < \alpha' < 1$.

Due to the lack of a characteristic time scale and in contrast to all previous cases, here the limiting distribution of R_t depends on the coupling between R_i and T_i, so two cases may be distinguished.

Case 1. R_i Is Independent of T_i. This relates to the decoupled CTRW. One may apply Eq. (31) to Theorem 1 of Dobrusin[26] or Lemma 1 of Kesten[23] to obtain

$$\Pr\left(\frac{R_t}{t^{\alpha'/\alpha} c_7} \leqslant r\right) \to U(r), \qquad 0 < \alpha' < 1 \tag{36}$$

where the normalizing constant $c_7 = c_2 c_6^{1/\alpha}$ with c_2 and c_6 defined by Eqs. (16) and (31) if $0 < \alpha < 2$, while if $\alpha = 2$ then $C_7 = (\text{Var } R_i/2)^{1/2} \, C_6^{1/\alpha}$. The limiting distribution in Eq. (36) is

$$U(r) = \int_0^\infty S_{\alpha, \beta}(rx^{-1/\alpha}) \, H_{\alpha'}(dx) \tag{37}$$

or, equivalently, from Eq. (10),

$$U(r) = \int_0^\infty S_{\alpha, \beta}(rx^{\alpha'/\alpha}) \, S_{\alpha', 1}(dx) = \Pr\left(\frac{X}{Y^{\alpha'/\alpha}} \leqslant r\right) \tag{38}$$

where X, Y are independent random variables having the distribution functions $S_{\alpha, \beta}(r)$ and $S_{\alpha', 1}(r)$, respectively.

Case 2. R_i Depends on T_i. This relates to a CTRW with a coupled memory. We give a specific example of the coupled memory in which the limiting distribution of R_t differs from the decoupled case, namely we assume $R_i = T_i$ and apply Theorem 1, ref. 9, XIV.3, to obtain as $t \to \infty$

$$\Pr\left(\frac{R_t}{t} \leqslant r\right) = \Pr\left(t^{-1} \sum_{i=0}^{N_t} T_i \leqslant r\right) \to W(r) \tag{39}$$

where the distribution $W(r)$ is concentrated on the interval (0, 1) and has the density $w(r) = [(\sin \pi\alpha')/\pi](1 - r)^{-\alpha'} r^{\alpha' - 1}$, $0 < r < 1$.

Let us consider the decoupled memory counterpart of Eq. (39) and suppose that R_i has the same distribution as T_i, but R_i is independent of T_i; thus, using $\alpha = \alpha'$, $\beta = 1$ in Eqs. (36) and (38) directly leads to $c_7 = 1$ and

$$\Pr(R_t/t \leqslant r) \to \int_0^\infty S_{\alpha', 1}(rx) \, S_{\alpha', 1}(dx) \tag{40}$$

as $t \to \infty$. Thus the limiting distribution in the decoupled case, Eq. (40), is concentrated on the half-line $(0, \infty)$ (see also ref. 27) and differs from the limit in the coupled case, Eq. (39).

4. CONCLUSIONS

1. In this paper we provide a systematic analysis of the asymptotic behavior of the distance R_t reached at time t for the coupled and decoupled memory CTRW in one dimension.

2. In contrast to other work the present one is based on renewal theory and the limit theorems for random sum of jumps R_i instead of Tauberian theorems for the two-dimensional Laplace–Fourier transform.

3. The results obtained here are in agreement with the previous studies of the decoupled memory CTRWs and Lévy walks. They give insight into the class of possible limiting distributions, which turns out to be surprisingly small. Also, it is worth stressing that the transition from the decoupled memory to the coupled memory does not change the form of the limiting distribution of R_t, except for the case of unbiased walks with an infinite mean waiting time, which is described in Section 3.3.

4. This approach provides an analytical form for the normalizing constants $c_1,..., c_7$ in Table I, which can be used for computer simulations.

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